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Solution of the Steady Euler Equations in a Generalized Lagrangian Formulation

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A second-order Godunov-type shock-capturing scheme for solving the steady Euler equations in generalized Lagrangian coordinates has been developed and applied to compute steady supersonic and hypersonic flow problems. Following Hui and Zhao, the Lagrangian distance and a stream function are used as the coordinate lines that not only simplify the Riemann solution procedure but also have an intrinsic flow adaptive property embedded. Numerical examples for various supersonic flows involving strong flow discontinuities are given. Good agreement is obtained between computed results and shock expansion theory or available experimental data. It was found that the resolution of the slip line is almost exactly without smearing, the resolution of shock is always crisp even at increasing Mach number, and the Prandtl-Meyer expansion is adequately resolved with the second-order-accurate scheme.

I. Introduction

RECENTLY, a class of new Lagrangian formulation of the Euler equations of gasdynamics has been proposed by Hui et al.[1-3] as an alternative way to describe the equations governing the inviscid compressible flows. For two-dimensional supersonic flow, the Lagrangian time τ and a stream function ξ are used as independent variables in the new Lagrangian description instead of the usual Cartesian coordinates (x, y) in the Eulerian description. For supersonic flows the steady Euler equations of gasdynamics are hyperbolic; hence, either the Godunov method[4] based on the exact Riemann solver or approximate Riemann solvers can be constructed to solve them.

The Godunov method for solving the steady supersonic flows consists of employing the self-similar solution of a steady Riemann problem involving an oblique shock wave, a Prandtl-Meyer expansion, and a slip line (tangential discontinuity). This method has been given by Glaz and Wardlaw[5] for the Eulerian formulation and by Loh and Hui[1] for the (τ, ξ) formulation. A flux difference splitting method for directly solving the steady Euler equations for supersonic flow has been given by Pandolfi.[6] In Ref. 2, a second-order total variation diminishing (TVD) scheme[7] has been devised for the new Lagrangian (τ, ξ) formulation by applying Sweby's flux limiter[8] in a scalar manner, which results in a rather restricted Courant number; i.e., a very small Courant number must be used. This may be attributed to the physical quantities used in the limiter functions. Nevertheless, it has been successfully applied to a wide variety of supersonic and hypersonic flow problems.

Several desirable features of the new Lagrangian method have been illustrated regarding the resolution of flow discontinuities in high-speed compressible flows. In particular, it is shown that 1) it resolves slip line crisply; 2) its accuracy improves with increasing Mach number due to the intrinsic flow adaptive nature; and 3) it requires no grid generation, yet the flow tangency condition at the solid boundary is automatically satisfied.

Despite the aforementioned features it was found that deficiencies exist in the Lagrangian time-stream function (τ, ξ) formulation: 1) The flux function is discontinuous across a slip line (a cell interface in the new Lagrangian method), which not only makes the Godunov numerical flux at the cell interface discontinuous but also complicates the construction of high-order schemes; 2) the system is not fully hyperbolic (see Ref. 9), in the sense that, although there exist six real eigenvalues, there are only five linearly independent eigenvectors associated with them; and 3) the system using the Lagrangian time is difficult to apply to subsonic flow problems.

To remove the aforementioned deficiencies of the Lagrangian time-stream function (τ, ξ) formulation, a generalized Lagrangian formulation using Lagrangian distance and stream function (λ, ξ) as independent variables has been proposed recently by Hui et al.,[10] who show that the generalized Lagrangian formulation of the steady Euler equations is fully hyperbolic for supersonic flow and the flux vector is continuous at the interface of a Godunov numerical flux.

The purpose of this study is first to devise a more robust second-order Godunov-type scheme for the two-dimensional steady Euler equations in generalized Lagrangian formulation. The scheme is conservative (expressed in the form of a numerical flux) and has the full linear stability bound based on the eigenvalues of the generalized Lagrangian system. First, the first-order Godunov scheme is extended to achieve high-order accuracy by adopting a Hancock[11] two-step procedure and using a van Leer's MUSCL[12] type approach together with an essentially nonoscillatory (ENO) interpolation.[13] The generalized Lagrangian formulation, by virtue of its use of streamlines and Lagrangian distance lines as coordinate lines, is expected to provide a basis for better representation of the tangential discontinuity as well as to follow the physics of fluid motion more closely. This expectation turns out to be well confirmed, and the generalized Lagrangian formulation, together with the Godunov scheme and its higher-order TVD and ENO schemes, has the potential of accurately and efficiently simulating steady supersonic and hypersonic flows in aerodynamics. Second, we apply the second-order Godunov-type schemes for the generalized Lagrangian Euler equations to simulate several supersonic flow problems and compare them with exact solutions, results from linearized theory and shock expansion theory, and available experimental data.

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In Sec. II the steady two-dimensional Euler equations in the generalized Lagrangian coordinates (λ, ξ) in conservation law form are described. In Sec. III the first-order Godunov scheme and its higher-order TVD and ENO extensions based on the Hancock two-step procedure and a MUSCL type approach are given. Several examples are computed in Sec. IV and compared with the exact solutions and available experimental results. Some practical supersonic and hypersonic flow problems are also included. Discussions of the generalized Lagrangian method and concluding remarks are given in Sec. V.

II. Generalized Lagrangian Euler Equations of Gasdynamics

We briefly describe the generalized Lagrangian formulation, introduced by Hui and Zhao[9] and Hui et al.[10] for the steady two-dimensional Euler equations. The conservation equations of mass, momentum, and energy for the steady two-dimensional inviscid compressible flows in the Cartesian coordinates can be expressed as

$$\frac{\partial \bar{E}}{\partial x} + \frac{\partial \bar{F}}{\partial y} = 0 \tag{1a}$$

where

$$\bar{E} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uH \end{bmatrix}, \qquad \bar{F} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vH \end{bmatrix} \tag{1b}$$

where ρ is the fluid density; u and v are the velocity components in the x and y directions, respectively; p is the pressure; and H is the total enthalpy. The pressure is related to other flow variables by the equation of state

$$H = \frac{1}{2}(u^2 + v^2) + \frac{\gamma p}{(\gamma - 1)\rho} \tag{2}$$

where $\gamma = c_p/c_v$ is the ratio of specific heats. Introducing a class of Lagrangian-type transformation of coordinates from (x, y) to (λ, ξ) as follows:

$$\mathrm{d}x = \frac{u}{q^\alpha}\,\mathrm{d}\lambda + U\,\mathrm{d}\xi \tag{3a}$$

$$\mathrm{d}y = \frac{v}{q^\alpha}\,\mathrm{d}\lambda + V\,\mathrm{d}\xi \tag{3b}$$

with

$$\frac{\partial x}{\partial \lambda} = \frac{u}{q^\alpha}, \qquad \frac{\partial x}{\partial \xi} = U \tag{4a}$$

$$\frac{\partial y}{\partial \lambda} = \frac{v}{q^\alpha}, \qquad \frac{\partial y}{\partial \xi} = V \tag{4b}$$

where α is a constant, and $q = (u^2 + v^2)^{1/2}$.

Two cases of α are of particular interest:

When $\alpha = 0$, one has

$$\mathrm{d}x = u\,\mathrm{d}\lambda, \qquad \mathrm{d}y = v\,\mathrm{d}\lambda, \quad \mathrm{d}x^2 + \mathrm{d}y^2 = q^2\,\mathrm{d}\lambda^2 \tag{5}$$

along a streamline $\xi = \text{const}$. Evidently, the variable λ so introduced is the time of motion of a particle along its streamline (path) and is thus the Lagrangian time τ and the system is denoted as the (τ, ξ) formulation.

When $\alpha = 1$, one has

$$\mathrm{d}x = \frac{u}{q}\,\mathrm{d}\lambda, \qquad \mathrm{d}y = \frac{v}{q}\,\mathrm{d}\lambda, \qquad \mathrm{d}x^2 + \mathrm{d}y^2 = \mathrm{d}\lambda^2 \tag{6}$$

along a streamline $\xi = \text{const}$ and a variable λ so defined in Eqs. (3) is clearly the distance traveled by a particle along its streamline (hence the name "Lagrangian distance" introduced by Hui and Zhao[9]), and the system is denoted as the (λ, ξ) formulation.

Integrating Eqs. (1) over an arbitrary domain and applying the Gauss divergence theorem and with the Lagrangian-type transformation defined by Eqs. (3), one can obtain the following equations of motion in the (λ, ξ) formulation in conservation law form:

$$\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial \xi} = 0 \tag{7a}$$

where

$$E = \begin{bmatrix} K \\ Ku + pV \\ Kv - pU \\ KH \\ U \\ V \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ -p \sin\theta q^{(1-\alpha)} \\ p\cos\theta q^{(1-\alpha)} \\ 0 \\ -\cos\theta q^{(1-\alpha)} \\ -\sin\theta q^{(1-\alpha)} \end{bmatrix} \tag{7b}$$

Here, $\cos\theta = u/q$, $\sin\theta = v/q$, and θ is the flow inclination angle. In the system (7), K is defined by

$$K(\xi) = \rho(uV - vU) \tag{8}$$

Instead of having four equations to solve in the Eulerian formulation, here one has six equations in the generalized Lagrangian formulation. The first four equations in Eq. (7b) can, of course, also be derived directly from the physical laws of conservation of mass, momentum, and energy, respectively, whereas the last two equations arise from the compatibility conditions between the λ derivatives of x and y, which contribute to the intrinsic flow adaptive nature of the generalized Lagrangian method.

For supersonic flows, the system (7) has real eigenvalues

$$\sigma_0 = 0, \qquad \text{multiplicity of 4} \tag{9a}$$

and

$$\begin{aligned}\sigma_\pm &= \frac{q^{1-\alpha}}{T}\,\frac{\cos(\beta-\theta) \pm \sin(\beta-\theta)\sqrt{M^2-1}}{M^2\sin^2(\beta-\theta) - 1} \\ &= \frac{q^{1-\alpha}\sin\mu}{T\sin[\pm(\beta-\theta) - \mu]}\end{aligned} \tag{9b}$$

where

$$\begin{aligned} U = T\cos\beta, &\qquad V = T\sin\beta \\ \mu = \sin^{-1}(1/M) &= \sin^{-1}(c/q)\end{aligned} \tag{10}$$

and where $c = (\gamma p/\rho)^{1/2}$ is the speed of sound, M is the Mach number, and $T = (U^2 + V^2)^{1/2}$.

It was noted in Ref. 9 that, within the class of Lagrangian-type transformations (3), the resulting equations can be shown to be fully hyperbolic if and only if $\alpha = 1$, i.e., when λ represents the distance traveled by a particle along its streamline. For $\alpha \neq 1$ (including the special case $\alpha = 0$ when λ represents the Lagrangian time τ), only five linearly independent left eigenvectors exist and the system is not fully hyperbolic in the sense that, although six real eigenvalues exist, there are only five linearly independent eigenvectors associated with them. Although the lack of a complete set of linearly independent eigenvectors does not in any way hinder the computational scheme based on marching in τ using the Godunov method (see Refs. 1 and 2), it is desirable to make a transformation to render the system truly hyperbolic so that many characteristic-based upwind schemes such as flux difference splitting[14] and flux vector splitting[15,16] may be applied to the generalized Lagrangian formulation.

Equations (7) in conservation form with $\alpha = 1$ will be used in Sec. III to construct Godunov-type schemes based on the elementary solutions of the steady Riemann problem. A key element of the Godunov method is to construct the numerical flux at the cell interface between two interacting cells. The exact solution of the steady Riemann problem yields the pressure p^* and the flow inclination angle θ^* at the cell interface, which happens to be a slip line in the generalized Lagrangian method. Across such a slip line (cell interface), the flux vector F of Eqs. (7) is discontinuous in the case of $\alpha = 0$ [i.e., the (τ, ξ) formulation]. This is so because the tangential component of velocity and hence the flow speed $q = (u^2 + v^2)^{1/2}$ is discontinuous, whereas the pressure p and the flow inclination angle $\theta = \tan^{-1}(v/u)$ are continuous. When a marching scheme (in τ) is used, the numerical fluxes on opposite sides of the boundary $\xi = \text{const}$ of neighboring cells do not cancel out after summing together, failing to satisfy the sufficient condition in the Lax-Wendroff theorem.[17] Consequently, there is no guarantee of capturing the slip line correctly. In practical computation for two-dimensional flow, this deficiency is easily corrected by adjusting the marching Lagrangian time steps for the neighboring cells so that they are inversely proportional to the flow speeds. Another disadvantage of the Lagrangian time and stream function formulation is that the extension of the first-order Godunov scheme to high resolution schemes such as TVD and ENO schemes can become quite cumbersome due to the discontinuity of flux at the cell interface (see Ref. 18).

The following remarks of Ref. 9 are interesting regarding the system (7) with the (τ, ξ) formulation and the (λ, ξ) formulation. In a computational scheme that solves the (τ, ξ) system by marching in the Lagrangian time τ with same $\Delta\tau$ for all cells at each marching step, the computational cells behave exactly as fluid particles because τ is the true time of motion. In this situation, marching in τ means literally following the particles, and in this sense the formulation is fully Lagrangian. In contrast, in a computational scheme that solves the (λ, ξ) system by marching in the Lagrangian distance λ with same $\Delta\lambda$ for all cells at each marching step, the computational cells do not behave identically with the fluid particles: They march in the flow directions of the particles but not necessarily with their speeds. The (λ, ξ) formulation is therefore not fully Lagrangian and is termed the "generalized Lagrangian formulation." Extension of the two-dimensional Eqs. (7) to three-dimensional steady Euler equations can be readily done[9] by introducing an additional stream function η and following the same procedure just described. It leads to a system of 11 equations: 5 conservation equations and 6 compatibility conditions that govern the deformation of fluid particles. It is also noted that the tangency condition on a solid boundary $\xi = \xi_0$ can be prescribed in terms of λ, which is the arc length of the body shape regardless of whether the flow is subsonic or supersonic. This is because the body shape can be described parametrically in terms of its arc length λ. It is possible to solve subsonic flow using the (λ, ξ) formulation, but methods other than space-marching methods have to be used. In contrast, if the Lagrangian time τ is used, there is no a priori way of prescribing the boundary conditions in terms of τ, which, in the subsonic flow case, depend on the solution to the flow, whence the mapping $x(\tau, \xi_0)$, $y(\tau, \xi_0)$ is not known a priori.

In the present work we will concentrate on the case of $\alpha = 1$, the Lagrangian distance and stream function (λ, ξ) formulation in conservation law form. A second-order Godunov-type scheme having a TVD or ENO property will be given later to solve the system (7) for fully supersonic flows. The restriction to fully supersonic flow is due to the numerical method adopted here, since we employ space-marching upwind schemes.

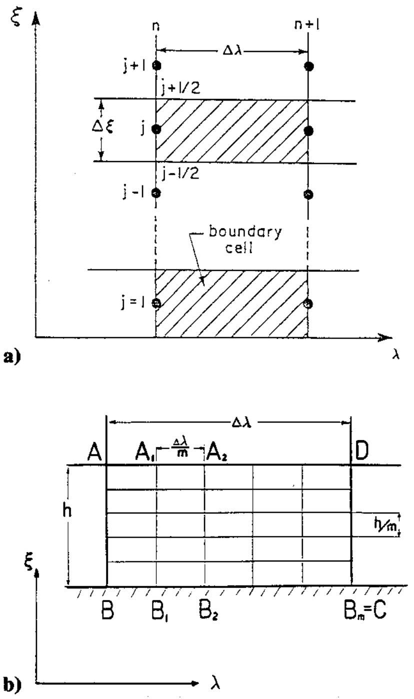

Fig. 1 Generalized Lagrangian coordinates (λ, ξ): a) computational mesh; b) boundary subcells.

III. Second-Order Nonoscillatory Godunov-Type Scheme

To compute a flow solution to Eqs. (7), a rectangular grid system is used to cover the flow domain in the λ-ξ plane and the computation marches in the increasing λ direction (Fig. 1a). Let the superscript n refer to the marching step number and the subscript j refer to the cell number. The marching Lagrangian distance step $\Delta\lambda^n = \lambda^{n+1} - \lambda^n$ is uniform for all j. It may vary with n but is always chosen to satisfy the usual linear stability condition based on the eigenvalues given in Eqs. (9). The mesh divides the computational domain into control volumes or cells that in the ξ direction are centered at (λ^n, ξ_j) and have a size of $\Delta\xi_j = \xi_{j+1/2} - \xi_{j-1/2}$ (for all n).

The difference equations for the jth cell at marching step n are formally derived by integrating (7) over the shaded rectangle in Fig. 1 and applying the divergence theorem. The result is

$$E_j^{n+1} = E_j^n - \frac{\Delta\lambda^n}{\Delta\xi_j}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right) \tag{11}$$

Here, for any quantity f,

$$f_j^n = \frac{1}{\Delta\xi_j}\int_{\xi_{j-1/2}}^{\xi_{j+1/2}} f(\lambda^n, \xi)\, \mathrm{d}\xi \tag{12}$$

is the cell average of f, and

$$f_{j+1/2}^{n+1/2} = \frac{1}{\Delta\lambda^n}\int_{\lambda^n}^{\lambda^{n+1}} f(\lambda, \xi_{j+1/2})\, \mathrm{d}\lambda \tag{13}$$

is the distance average.

In the first-order Godunov scheme, the jth cell average, E_j^n at the marching step n is considered as constant within that cell and the flux $F_{j+1/2}^{n+1/2}$ along the interface (a streamline) between the jth cell and the $(j+1)$th cell from marching step n to $n+1$ is to be obtained from the self-similar solution $R[(\xi - \xi_{j+1/2})/(\lambda - \lambda^n);\ Q_j^n;\ Q_{j+1}^n]$ at $\xi = \xi_{j+1/2}$ to the Riemann problem formed by these two adjacent constant flow states Q_j^n and Q_{j+1}^n, where $Q = (\rho, u, v, p)^T$. This flux will be denoted with a superscript G [i.e., $F_{j+1/2}^{n+1/2} = F_{j+1/2}^G$]. The solution to the Riemann problem yields a flow consisting of an oblique shock wave, a Prandtl-Meyer expansion, and a slip line (tangential discontinuity). However, no special consideration is needed for slip lines, since they are also streamlines in steady flow. This yields a much simpler Riemann solution procedure and is one of the advantages of the generalized Lagrangian formulation.

In practical application, the first-order Godunov scheme is not accurate enough and needs to be upgraded to higher-order accuracy. A first-order upwind scheme such as the Godunov scheme [Eq. (11)] can be upgraded to a second-order method by first advancing the cell boundary values to the intermediate marching level $\lambda^{n+1/2} = \lambda + \Delta\lambda/2$. In obtaining these values the interaction between cells can be fully ignored. This observation, due to Hancock, has led to a simpler implementation of second-order upwind schemes.[11] In the following, such a two-step procedure is adopted and extended to construct a second-order nonoscillatory Godunov-type scheme for solving Eqs. (7) using an ENO interpolation. The scheme can be written as

$$E_j^{n+1} = E_j^n - \frac{\Delta\lambda^n}{\Delta\xi_j}[F_{j+1/2}^N - F_{j-1/2}^N] \tag{14}$$

where $F_{j+1/2}^N$ is the numerical flux defined by

$$F_{j+1/2}^N = F^G(E_{j+1/2}^{*+}, E_{j+1/2}^{*-}) \tag{15}$$

Here, $E_{j\pm1/2}^{*\mp}$ are intermediate values given by

$$E_{j+1/2}^{*-} = E_{j+1/2}^{n-} - \frac{\Delta\lambda^n}{2\Delta\xi_j}[F(E_{j+1/2}^{n-}) - F(E_{j-1/2}^{n+})] \tag{16a}$$

$$E_{j-1/2}^{*+} = E_{j-1/2}^{n+} - \frac{\Delta\lambda^n}{2\Delta\xi_j}[F(E_{j+1/2}^{n-}) - F(E_{j-1/2}^{n+})] \tag{16b}$$

In Eqs. (16), $E_{j+1/2}^{n-}$ and $E_{j-1/2}^{n+}$ are more accurate values at interfaces using ENO interpolation[13] and are given by

$$E_{j+1/2}^{n-} = E_j^n + \tfrac{1}{2}S_j^n \tag{17a}$$

$$E_{j-1/2}^{n+} = E_j^n - \tfrac{1}{2}S_j^n \tag{17b}$$

with

$$S_j^n = m[\Delta_+E_j^n - \zeta\bar{m}(\Delta_-\Delta_+E_j^n, \Delta_+\Delta_+E_j^n), \Delta_-E_j^n + \zeta\bar{m}(\Delta_+\Delta_-E_j^n, \Delta_-\Delta_-E_j^n)] \tag{18}$$

where $\Delta_\pm E_j = \pm(E_{j\pm1} - E_j)$ denote the usual forward and backward difference operators.

The limiter functions m and $\bar{m}$ are defined by

$$m(a, b) = \begin{cases} s\min(|a|, |b|) & \text{if} \quad \operatorname{sgn} a = \operatorname{sgn} b = s \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

and

$$\bar{m}(a, b) = \begin{cases} a, & \text{if} \quad |a| \le |b| \\ b, & \text{if} \quad |a| > |b| \end{cases} \tag{20}$$

The scheme defined above by Eqs. (14) is stable if

$$\sigma_{\max}\Delta\lambda/\Delta\xi \le 1 \tag{21}$$

where $\sigma_{\max}$ is the maximum value of the eigenvalues given in Eqs. (9). It can also be shown that the scheme is of second-order accuracy in both $\Delta\lambda$ and $\Delta\xi$ for the scalar wave equation using Taylor expansion.

If $\zeta = 0$, one has a second-order TVD scheme, and if $\zeta = 1/2$, one has a uniformly second-order ENO scheme.

It is noted that the limiter functions in Eq. (18) are placed on quantities of flux nature ($\Delta_\pm E$). This avoids the use of characteristic variables that can be very expensive in the generalized Lagrangian method due to the large matrix system involved. The present algorithm is quite economic and works successfully in practical computations. Other flux limiters such as those given in Ref. 8 and the one in Ref. 19 using a third-order nonoscillatory interpolation due to Harten et al.[20] can also be employed to yield high-order schemes.

Finally, after updating the solution, we have to decode E_j^{n+1} to get Q_j^{n+1} and thus complete the procedure of marching forward in λ by one step. The decoding is rather straightforward. Let

$$A = \frac{1+\gamma}{1-\gamma}(e_5^2 + e_6^2)$$

$$B = \frac{-2}{\gamma - 1}(e_2e_6 - e_3e_5)$$

$$C = e_2^2 + e_3^2 - 2K^2H$$

Then the pressure p satisfies the equation

$$Ap^2 + Bp + C = 0$$
$$p = [-B + \sqrt{B^2 - 4AC}]/2A \tag{22}$$

The plus sign is chosen[1] because otherwise negative pressure may result. Once the pressure p is obtained, then the other flow variables are

$$u = (e_2 - e_6p)/K \tag{23}$$

$$v = (e_3 + e_5p)/K \tag{24}$$

$$\rho = K/(ue_6 - ve_5) \tag{25}$$

At this stage the numerical procedure is completed.

If a solid boundary is present in the flow, it must be a streamline and hence can be identified by $\xi = \xi_0 = $ const, on which the inclination angle θ is given. The boundary condition to be imposed on the solid boundary is then

$$v/u = \tan\theta_w \tag{26}$$

where θ_w is the inclination of the solid boundary. This results in a boundary Riemann problem and is solved using the method described in Ref. 1. This is only first-order accurate. For supersonic flow past a solid body, the body shape determines the development of the subsequent flow. It is therefore crucial to be able to compute the flow near the solid boundary accurately. A subcell treatment[2] for the boundary cell is described as follows. Let the cell adjacent to the wall be denoted by $j = 1$. To obtain an improved E_1^{n+1}, the width h of the original cell AB (Fig. 1b) at integration step n is divided into m equal subcells of width h/m. The first-order Godunov scheme is then applied to compute the evolution of these subcell flows by marching, with submarching step $\Delta\lambda/m$, $\Delta\lambda$ being the original Lagrangian distance step. In doing so the interface flux originally calculated along the cell upper boundary AD is used as the flux along AA_1, and the flux along the body surface of the slope of BB_1 is used (in the boundary Riemann solver) instead of the original slope of BC. This procedure is repeated m times until the original Lagrangian distance integration step $n+1$ is reached, during which the upper interface flux is held to be the same as given by the original flux along AD but the surface slope assumes its local values at the subsegments $BB_1, B_1B_2, \ldots, B_{m-1}B_m$. After the last integration step, E_{1i}^{n+1} $(i = 1, 2, \ldots, m)$ are obtained and the arithmetic average is then taken to yield

$$E_1^{n+1} = \frac{1}{m}\sum_1^m E_{1i}^{n+1} \tag{27}$$

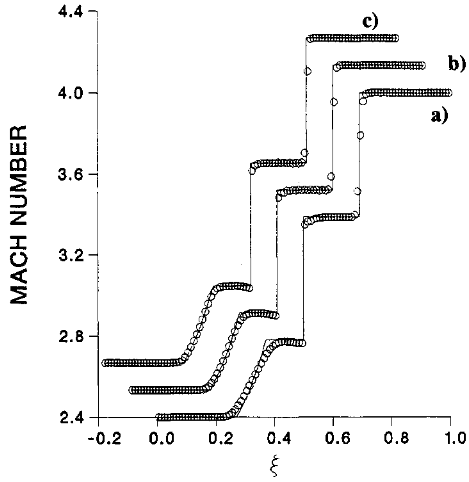

Fig. 2 Solution of steady Riemann problem for supersonic flow (Mach number distribution at λ = 0.5): comparison of schemes. Symbols, computation; solid lines, exact solution: a) first-order Godunov; b) TVD; c) ENO.

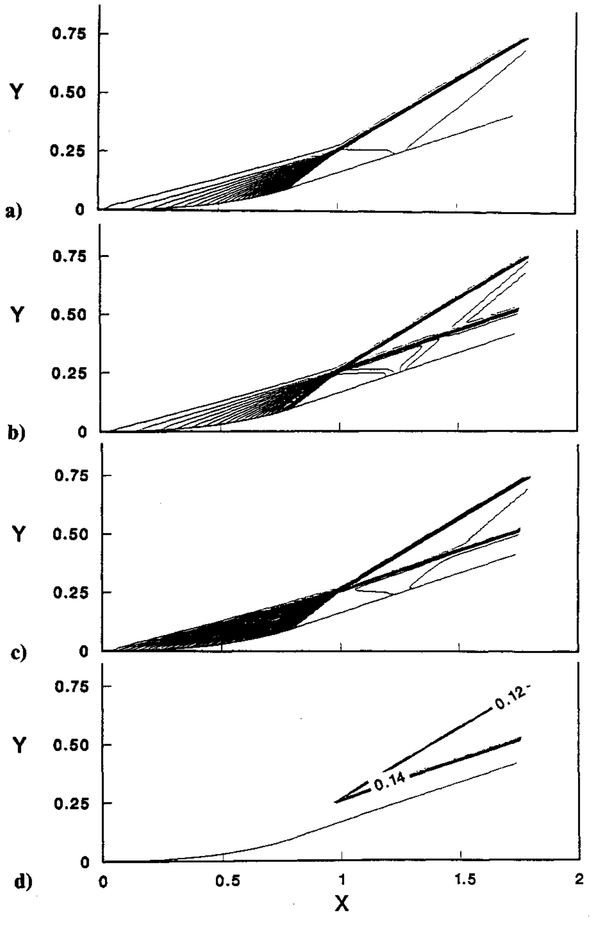

Fig. 3 Sudden formation of oblique shock wave of finite strength (M_∞ = 4.0, M_1 = 2.867); second-order TVD solution: a) pressure contours; b) density contours; c) Mach number contours; d) entropy contours.

This completes the subcell treatment for the boundary cell from integration step n to $n + 1$. The computation can then be marched as usual from integration step $n + 1$ to $n + 2$, again with subcell treatment for the boundary cell. The subcell treatment for the boundary cell is easy to implement and, with $m = \mathcal{O}(h^{-1/2})$, the accuracy of the computed flow in the boundary cell is found to be typically improved to the same level as the interior cells, making the computation uniformly second-order accurate.

IV. Numerical Results

In this section we apply the preceding second-order nonoscillatory schemes for the generalized Lagrangian Euler equations of gasdynamics to solve several initial boundary-value problems of supersonic flows and compare the results with the exact solutions. Computations of some practical supersonic and hypersonic flow problems are also included. In the following computations the exact Riemann problems are solved to a tolerance of 10^{-6}. With the intersection point of the two tangents at the initial states as initial guess, very little CPU time is needed to get to 10^{-6}.

First we consider an initial value steady Riemann problem formed by the confluence of two parallel supersonic streams at $\xi_0 = 0.5$ with different states. The initial conditions are given by

$$Q = \begin{cases} Q_T = (\rho_T, p_T, M_T, \theta_T) = (0.5, 0.25, 4.0, 0.0\ \text{deg}), & \xi > \xi_0 \\ Q_B = (\rho_B, p_B, M_B, \theta_B) = (1.0, 1.0, 2.4, 0.0\ \text{deg}), & \xi \le \xi_0 \end{cases}$$

Here we use 100 cells and $\Delta\xi = 0.01$. The computed Mach number distribution (symbols) at section $A - A'$ ($\lambda = 0.5$) is shown in Fig. 2 along with the exact solutions (solid line) that are generated using the Riemann solver. Results obtained using both first-order Godunov, second-order TVD, and ENO schemes with a Courant-Friedrick-Lewy (CFL) number of 0.95 are shown. All computations are performed on a Convex C1 vector computer. The CPU times are 1.32, 1.58 and 1.69 s, respectively, for the first-order Godunov, the second-order TVD, and the ENO schemes. Both the second-order TVD and ENO schemes can give very good resolution of the flow discontinuities, and the ENO result indicates slightly better resolution overall.

Second, we consider a supersonic flow $M_0 > 1$ past a concave wall for which a shock wave of finite strength is formed suddenly in the interior of the flowfield and consequently a slip line is also generated. The equation for the profile of sudden formation of a shock wave has been given in Ref. 2. As an example, we consider the case when $M_\infty = 4.0$ and $M_1 = 2.867$. The flowfield is computed using the second-order TVD

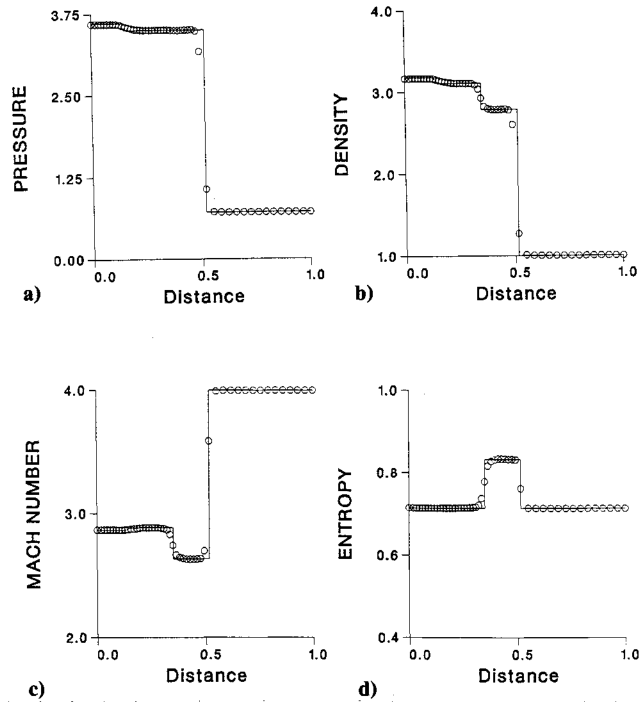

Fig. 4 Distributions of flow properties along a Lagrangian distance line (λ = 1.15): a) pressure; b) density; c) Mach number; d) entropy.

Table 1 Comparison of aerodynamic parameters for two-dimensional circular arc airfoil ($M_\infty = 2.0$, $t/c = 0.1$)

α, deg	C_l	C_d
0	0.000001	0.031108[a]
	0.000000	0.031229[b]
	0.000000	0.030916[c]
2	0.083950	0.034053
	0.083908	0.034274
	0.080613	0.033730
4	0.168524	0.043285
	0.168297	0.043464
	0.161227	0.042172
6	0.253870	0.058830
	0.253700	0.058980
	0.241840	0.056241
8	0.340307	0.080972
	0.340716	0.081155
	0.322453	0.075939
10	0.429523	0.110457
	0.429280	0.110426
	0.403067	0.101264

[a]Present computation. [b]Shock expansion theory. [c]Linearized theory.

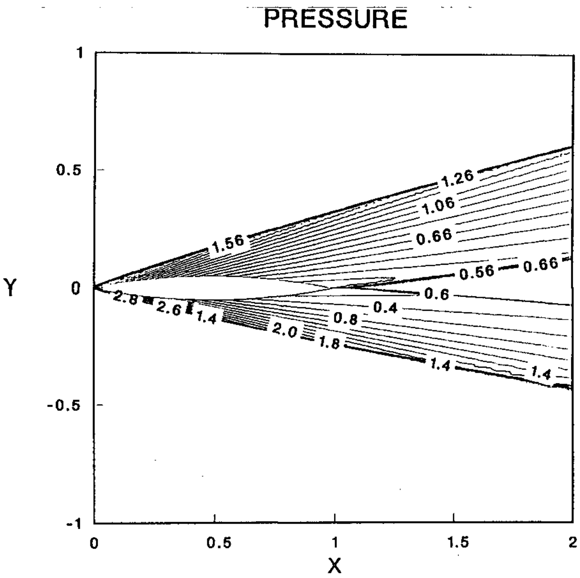

Fig. 5 Steady hypersonic flow around a symmetric circular arc airfoil. Pressure contours (upper, $M_\infty = 5.0$; lower, $M_\infty = 8.0$).

scheme with 75 cells and $\Delta\xi = 0.01$. In Fig. 3 the computed pressure contours, density contours, Mach number contours, and entropy contours are shown. The sudden birth of an oblique shock wave of finite strength in the interior of the flowfield and the accompanying Prandtl-Meyer expansion and the slip line are seen well captured. The distributions of flow properties along a Lagrangian distance line ($\lambda = 1.15$) are shown in Fig. 4 together with the exact solutions (solid lines).

We also consider supersonic flows over two-dimensional profiles. Two types of profiles are considered. One is a 10% thick ($t/c = 0.1$) symmetric circular arc airfoil, and the other is a symmetric double-wedge airfoil with a wedge angle of 10 deg. For these two airfoils, 100 uniform cells with $\Delta\xi = 0.01$ are used and CFL = 0.9. In Table 1, the computed lift coefficient C_l and wave drag coefficient C_d (top line) for supersonic flow over a symmetric circular arc airfoil ($t/c = 0.1$) with several angles of attack are listed together with results obtained using shock expansion theory (middle line) and linearized theory (bottom line). Excellent agreement is found between the present computation and the shock expansion theory. Results from the linearized theory deviate substantially from those of the nonlinear theory and the Euler computation at increasing angle of attack.

In Fig. 5 the pressure contours around a symmetric circular arc airfoil in a hypersonic flow at 0-deg angle of attack are shown together for $M_\infty = 5.0$ (upper) and $M_\infty = 8.0$ (lower), respectively. The leading-edge and trailing-edge oblique shock waves and the Mach waves emanating from the surface are all crisply resolved. For the $M_\infty = 8.0$ case the initial mesh distribution is stretched with $\Delta\xi_{\min} = 0.001$.

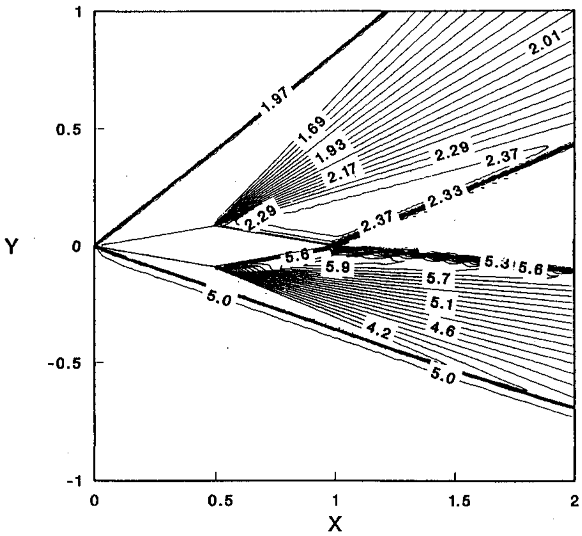

Fig. 6 Steady supersonic flow around a double-wedge airfoil. Mach number contours (upper, $M_\infty = 2.0$; lower, $M_\infty = 5.0$).

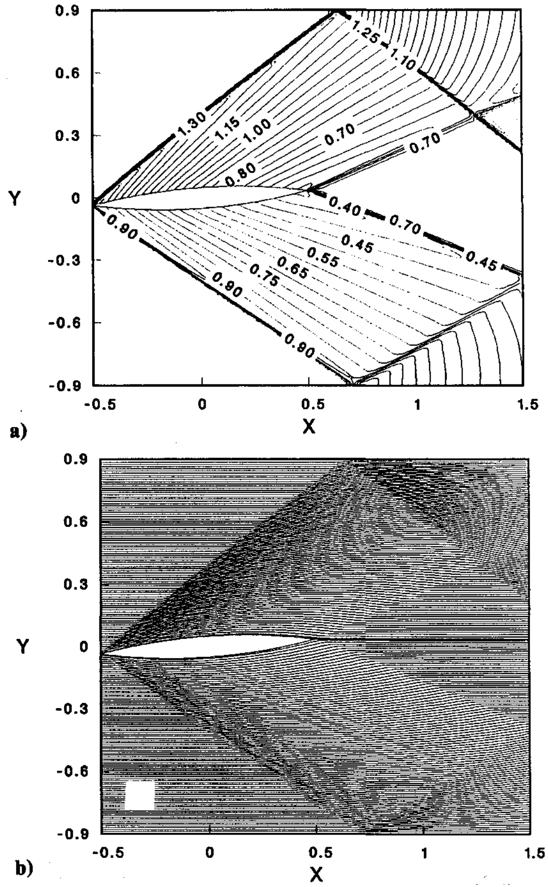

Fig. 7 Supersonic flow around a symmetric circular arc profile ($M_\infty = 2.05$; $\alpha = 4$ deg): a) pressure contours; b) particle traces (grid lines).

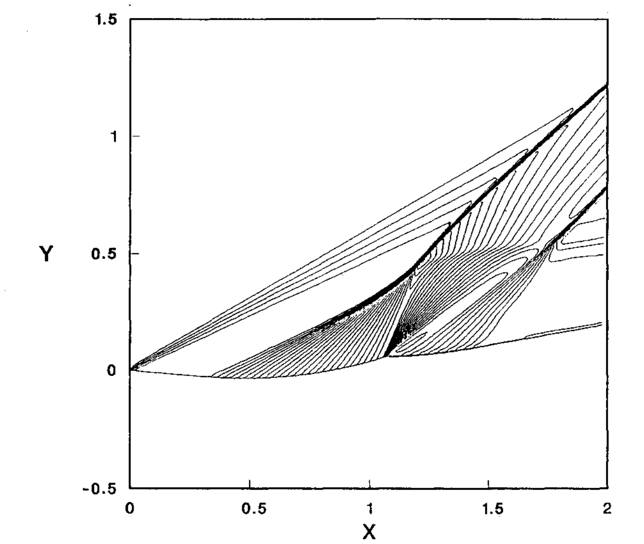

Fig. 8 Supersonic flow around a concave profile ($M_\infty = 1.87$); Mach number contours.

Computed results for supersonic flow past a symmetric double-wedge airfoil with a 10-deg wedge angle are given in Fig. 6 for freestream Mach number $M_\infty = 2.0$ (upper) and $M_\infty = 5.0$ (lower). Only the Mach number contours are shown. The leading-edge oblique shocks, the Prandtl-Meyer expansion fans, and the trailing-edge oblique shocks are all crisply represented.

For flow past the two-dimensional airfoils mentioned earlier, the CPU time for each case ranges from 22 to 25 s.

The next example is a supersonic flow $M_\infty = 2.05$ past a 10% symmetric circular arc profile with 4-deg angle of attack in a channel. The pressure contours and the resulting particle traces (grid lines) shown in Fig. 7 were obtained using the second-order TVD scheme. The number of cells used is 90, with $\Delta\xi = 0.01$ and CFL = 0.9. Good agreement is observed between the present computation and the experimental shadowgraph of Ferri.[21]

The last example we consider is an experimental case on supersonic flow with $M_\infty = 1.87$ past a concave wall shape consisting of two circular arc profiles used by Johannesen.[22] In this case there is a slope discontinuity at the junction of the two circular arc profiles. The computed Mach number contour is shown in Fig. 8. Good agreement is found between the computational and experimental results.[22]

V. Concluding Remarks

In this paper we have successfully applied the two-step procedure of Hancock and the ENO interpolation of Harten and Osher to yield a second-order nonoscillatory Godunov-type scheme for solving the steady Euler equations of gasdynamics in generalized Lagrangian formulation due to Hui and Zhao. Various supersonic and hypersonic flows are computed and compared with exact solutions and available experimental data. Very good agreement among the computational, theoretical, and experimental results has been obtained. The main features of the generalized Lagrangian formulation are 1) its use of the stream function ξ and the Lagrangian distance λ as independent variables, with the consequence that intrinsic flow adaption is embedded and the body surface boundary condition is satisfied exactly on a coordinate line without a grid generation; and 2) a slip line that is also a streamline and always coincides with the cell interface, which renders not only excellent resolution of slip lines but also much simpler and efficient implementation of the Godunov method compared with the Eulerian method. All of these generalized Lagrangian features and the second-order-accurate nonoscillatory Godunov-type scheme probably contribute to the results of the present work that oblique shock waves, Prandtl-Meyer expansions, and slip lines can be resolved crisply.

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